Denote $\eta(x) = P(Y = 1|X = x)$. Consider Bayes classifier with a reject option:

$$g_c(x) = \begin{cases} 1, & \eta(x) > 1/2 + c, \\ 0, & \eta(x) < 1/2 - c, \\ \text{r}, & \text{otherwise.} \end{cases} \tag{1}$$

Proposition 1. *For any decision $g(x)$ such that*

$$P(g(X) = r) \leq P(g_c(X) = r), \tag{2}$$

we have

$$P(g(X) \neq Y|g(X) \neq r) \geq P(g_c(X) \neq Y|g_c(X) \neq r). \tag{3}$$

In other words, the Bayes classifier $g_c(x)$ with a reject option is optimal for fixed rejection rate.

Proof. We first clarify some notations. Denote

$$A_c \equiv \{x| \max(\eta(x), 1 - \eta(x)) > 1/2 + c\} = \{x|g_c(x) \neq \text{r}\}, \tag{4}$$

which is the decision region. The error rate is

$$\begin{aligned} P(g_c(X) \neq Y|g_c(X) \neq \text{r}) =& 1 - P(g_c(X) = Y|g_c(X) \neq \text{r}) && (5) \\ =& 1 - \frac{P(g_c(X) = Y, g_c(X) \neq \text{r})}{P(g_c(X) \neq \text{r})} && (6) \\ =& 1 - \frac{1}{P(X \in A_c)} \int_{x \in A_c} \max(\eta(x), 1 - \eta(x)) dP, && (7) \end{aligned}$$

where dP is the measure of X, and $P(X \in A_c) = \int_{x \in A_c} dP$.

The strategy is to prove the following two lemmas:

Lemma 2. *Let B be any region such that $P(X \in B) \geq P(X \in A_c)$, then $P(g_c(X) \neq Y|X \in B) \geq P(g_c(X) \neq Y|X \in A_c)$.*

Lemma 3. *Let B be any region such that $P(X \in B) \geq P(X \in A_c)$ and g be any decision, then $P(g(X) \neq Y|X \in B) \geq P(g_c(X) \neq Y|X \in B)$.*

Lemma 3 is the same as proving the optimality of Bayes decision rule without rejection, hence we omit the proof. We are left to prove lemma 2.

Proof of Lemma 2. After arranging terms, it suffices to prove

$$P(X \in A_c)P(X \in B) [P(g_c(X) \neq Y|X \in B) - P(g_c(X) \neq Y|X \in A)] \geq 0, \tag{8}$$

where the L.H.S. of the inequality is

$$\text{L.H.S.} = P(X \in B) \int_{x \in A_c} \max(\eta(x), 1 - \eta(x)) dP - P(X \in A_c) \int_{x \in B} \max(\eta(x), 1 - \eta(x)) dP. \tag{9}$$

The key is to decompose $A_c = U + (A_c - U)$ and $B = U + (B - U)$, where $U = A \cap B$. By definition, for $x \in A_c - U$ or U, $\max(\eta(x), 1 - \eta(x)) > 1/2 + c$; for $x \in B - U$, $\max(\eta(x), 1 - \eta(x)) \leq 1/2 + c$. Thus

$$\int_{x \in A_c} \max(\eta(x), 1 - \eta(x)) dP \geq \left(\frac{1}{2} + c\right) P(X \in A_c - U) + \int_{x \in U} \max(\eta(x), 1 - \eta(x)) dP, \tag{10}$$

$$\int_{x \in B} \max(\eta(x), 1 - \eta(x)) dP \leq \left(\frac{1}{2} + c\right) P(X \in B - U) + \int_{x \in U} \max(\eta(x), 1 - \eta(x)) dP. \tag{11}$$

Insert Eq. (10) and (11) into (9),

$$
\begin{aligned}
\text{L.H.S.} \geq & \left(\frac{1}{2}+c\right)[P(X \in B)P(X \in A_c - U) - P(X \in A_c)P(X \in B - U)] \\
& + [P(X \in B) - P(X \in A_c)] \int_{x \in U} \max(\eta(x), 1-\eta(x)) dP && (12) \\
\geq & [P(X \in B) - P(X \in A_c)] \left[\int_{x \in U} \max(\eta(x), 1-\eta(x)) dP - \left(\frac{1}{2}+c\right) P(X \in U)\right] && (13) \\
\geq & 0, && (14)
\end{aligned}
$$

where in the last line we have used the assumption $P(X \in B) \geq P(X \in A_c)$. This proves lemma 2. □

Combine these two lemmas we establish the following inequality chain:

$$P(g(X) \neq Y | X \in B) \geq P(g_c(X) \neq Y | X \in B) \geq P(g_c(X) \neq Y | X \in A_c), \tag{15}$$

which proves the proposition. □

Remark. The proof of lemma 2 is essentially the proof of Neyman–Pearson lemma in statistics.