

8.08 Statistical Physics II — Spring 2019

Recitation Note 10

Huitao Shen
huitao@mit.edu

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1 Vortices in BEC

In the last recitation, we derived the GP equation for BEC with δ interaction:

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + (U(\mathbf{r}) - \mu) + g|\Phi(\mathbf{r})|^2 \right] \Phi(\mathbf{r}) = 0, \quad (1)$$

by minimizing the energy functional

$$H = \int d\mathbf{r} \Phi^*(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m} + (U(\mathbf{r}) - \mu) + \frac{g}{2} |\Phi(\mathbf{r})|^2 \right) \Phi(\mathbf{r}), \quad (2)$$

where $\Phi(\mathbf{r})$ is the complex-value order parameter with condensed boson density $\rho_0(\mathbf{r}) = |\Phi(\mathbf{r})|^2$.

1.1 Flow Quantization

Consider time-dependent GP equation, which can be obtained in a similar manner to time-independent GP equation (first apply Heisenberg equation of motion to field operators then apply Bogoliubov approximation):

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + (U(\mathbf{r}, t) - \mu) + g|\Phi(\mathbf{r}, t)|^2 \right] \Phi(\mathbf{r}, t). \quad (3)$$

Multiply Eq. (3) with Φ^* and subtract its complex conjugate, we have

$$\frac{\partial |\Psi|^2}{\partial t} + \frac{\hbar}{2mi} \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = 0. \quad (4)$$

Compare this equation with the continuity equation of hydrodynamics:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_0) = 0, \quad (5)$$

we can identify the condensate velocity field:

$$\mathbf{v}_0 = \frac{\hbar}{2mi} \frac{\Psi^* \nabla \Psi - \Psi \nabla \Psi^*}{|\Psi|^2}. \quad (6)$$

Since Φ is a complex field, we can always decompose it to a phase field and a modulus field: $\Phi = e^{i\phi} f$. It is straightforward to verify

$$\mathbf{v}_0 = \frac{\hbar}{m} \nabla \phi. \quad (7)$$

Therefore, the gradient of the phase of the order parameter describes the velocity field of the condensate.

It is straightforward to verify the velocity field is irrotational $\nabla \times \mathbf{v}_0 = 0$. With Stokes' theorem, this guarantees the circulation of the flow

$$\kappa = \oint \mathbf{v}_0 \cdot d\mathbf{r} = \frac{\hbar}{m} \Delta \phi, \quad (8)$$

does not depend on the path of the integral as long as it encircles the origin. However, $\Delta \phi$ must a multiple integer of 2π due to the single-valuedness of the order parameter, this implies the circulation

$$\kappa = \frac{h}{m} n, \quad (9)$$

is quantized. The above fact is called the flow quantization and n is called the winding number.

Flow quantization can be observed in many experiments. For example, one can stir normal helium-4 and then slowly cool down the system. Some rotating normal helium-4 will enter the condensate. However, the total circulation only jumps discontinuously, which are called phase-slip events.

1.2 Vortex

In the following, we assume the potential is uniform such that $U(\mathbf{r}) = 0$. We are going to look for a particular kind of ansatz to the GP equation with rotation symmetry and translation symmetry along z direction. In the cylindrical coordinates:

$$\Phi(\mathbf{r}) = e^{i\varphi n} f(r), \quad (10)$$

Here n is simply the winding number defined above. n is also the angular momentum. To see this, notice the velocity field of the ansatz is¹

$$\mathbf{v}_0 = \frac{\hbar}{m} \frac{n}{r} \hat{\varphi}. \quad (13)$$

The above ansatz with $n \neq 0$ is usually called a vortex. (Since we assume translation symmetry along z , it is actually a vortex line.) The velocity field diverges when $r \rightarrow 0$. This also means $\nabla \times \mathbf{v}_0 = 0$ does not hold at $r = 0$, which is called the vortex core. The reason is that the phase is not well-defined at the vortex core. In order for the kinetic energy to be finite, the condensate density and hence the order parameter must be zero at the vortex core.

We now try to solve time-independent GP equation with the ansatz Eq. (10). The GP equation becomes

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) - \frac{n^2}{r^2} f \right] + g f^3 = \mu f. \quad (14)$$

This differential equation can only be solved numerically. We can still analytically analyze the behavior of the solution.

First consider the large r limit, where the kinetic energy is insignificant. In this case, the equation becomes

$$f_0 = \sqrt{\frac{\mu}{g}}, \quad (15)$$

which is identical to the uniform ground state solution. On the other hand, when r is small, the kinetic energy dominates. One can verify the ansatz $f \sim r^n$ ensures the kinetic energy does not diverge when $r = 0$.

To find the typical size of the vortex, we change the variable $x = r/\xi$ and $\chi = f/f_0$, after which the equation becomes

$$-\frac{\hbar^2}{2mg\xi^2} \left[\frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \chi}{\partial x} \right) - \frac{n^2}{x^2} \chi \right] + f_0^2 \chi^3 = f_0^2 \chi. \quad (16)$$

If we let

$$\chi = \sqrt{\frac{\hbar^2}{2mgf_0^2}} = \sqrt{\frac{\hbar^2}{2mg\rho_0}}, \quad (17)$$

the equation becomes dimensionless:

$$\frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \chi}{\partial x} \right) - \frac{n^2}{x^2} \chi + \chi^3 = \chi. \quad (18)$$

ξ is called the correlation length or the healing length of the condensate, it describes the typical size of a vortex.

¹Here are two useful formulae:

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \varphi} \hat{\varphi} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}, \quad (11)$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}. \quad (12)$$

We can estimate the energy of the vortex. Suppose the size of the condensate is $R \gg \xi$. The energy of the vortex is

$$H = \int d^2\mathbf{r} \left\{ -\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) - \frac{n^2}{r^2} f \right] f - \mu f^2 + g f^4 \right\} \quad (19)$$

$$= 2\pi \int_0^R dr \left\{ -\frac{\hbar^2}{2m} \left[\frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) - \frac{n^2}{r} f \right] f - \mu r f^2 + g r f^4 \right\} \quad (20)$$

$$\approx 2\pi \int_\xi^R dr \left\{ -\frac{\hbar^2}{2m} \left[\frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) - \frac{n^2}{r} f \right] f - \mu r f^2 + g r f^4 \right\} \quad (21)$$

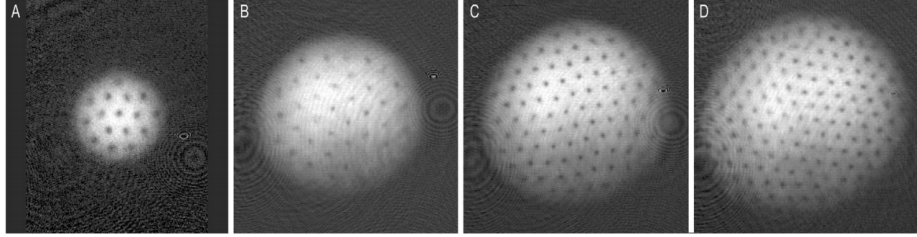
$$\approx 2\pi \int_\xi^R dr \frac{\hbar^2 n^2}{2m r} f_0^2 \quad (22)$$

$$= \frac{\pi \hbar^2 n^2 \rho_0}{m} \ln \left(\frac{R}{\xi} \right), \quad (23)$$

where we have neglected the energy contribution from the vortex core and approximated the order parameter outside ξ as a constant. The energy of the vortex diverges logarithmically with condensate size. This result in turn justifies our approximation of neglecting the vortex core contribution when R is large, because that part does not scale with R . It is confirmed numerically that the vortex energy indeed diverges logarithmically.

Finally, by taking interaction between vortices into account, these vortices tend to form into well ordered triangular lattice. This is called Abrikosov lattice and is observed in cold atom experiments [1].

Fig. 1. Observation of vortex lattices. The examples shown contain approximately (A) 16, (B) 32, (C) 80, and (D) 130 vortices. The vortices have "crystallized" in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20. Slight asymmetries in the density distribution were due to absorption of the optical pumping light.



2 Fermi-Dirac Distribution

Fermi-Dirac distribution is very similar to the chemical potential of molecules on a surface.

1. Different sites on the surface are non-interacting.
2. Each site can only host either 0 or 1 molecule, which reflects the Pauli exclusion principle of fermions.
3. The "site" becomes the "momentum": $n_i \rightarrow n_{\mathbf{k}}$.
4. The molecules on each site only has a site-dependent binding energy $\varepsilon_{\mathbf{k}}$. Other than that, the molecules on each site are non-interacting: $E_{n_{\mathbf{k}}}^{(i_{\mathbf{k}})} = n_{\mathbf{k}} \varepsilon_{\mathbf{k}}$.

In this way,

$$\Xi_{\mathbf{k}} = \sum_{n=0}^1 e^{-\beta n(\varepsilon_{\mathbf{k}} - \mu)} = 1 + e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}, \quad (24)$$

and the occupation number of fermions with a specific momentum:

$$\langle n_{\mathbf{k}} \rangle = \frac{\partial \ln \Xi_{\mathbf{k}}}{\partial(\beta\mu)} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} + 1}. \quad (25)$$

This is the so-called Fermi-Dirac distribution.

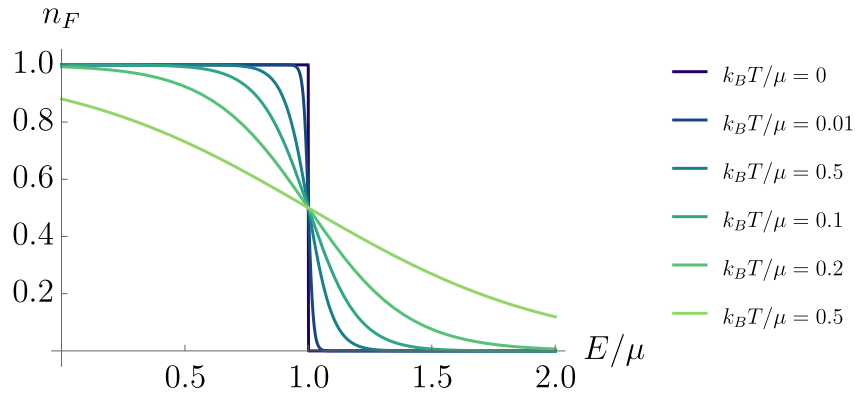


Figure 1: Fermi-Dirac distribution of various temperatures.

It is very helpful to visualize Fermi-Dirac distribution. At strictly zero temperature, the Fermi-Dirac distribution becomes a step function, which is obviously the minimal energy distribution. At low temperature, states with energy below the chemical potential μ is mostly occupied, and states with energy above the chemical potential μ is mostly empty. Roughly speaking, at temperature T , $k_B T / \mu$ states are excited to have $k_B T$ more energy. Therefore,

$$U(T) - U(0) \sim \frac{k_B T}{\mu} k_B T, \quad (26)$$

and the heat capacity is linear

$$C_V(T) = \frac{\partial U(T)}{\partial T} \sim T. \quad (27)$$

References

- [1] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle. Observation of Vortex Lattices in Bose-Einstein Condensates. *Science*, 292(5516):476–479, 2001.