8.08 Statistical Physics II — Spring 2019
Recitation Note 11

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Contents

1 Non-interacting Fermi Gas

For non-interacting Fermi gas, we have the grandpartition function

$$\Xi_{\mathbf{k}} = \sum_{n=0}^{1} e^{-\beta n(\varepsilon_{\mathbf{k}} - \mu)} = 1 + e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}, \tag{1}$$

and the Fermi-Dirac distribution of particle numbers:

$$\langle n_{\mathbf{k}} \rangle = \frac{\partial \ln \Xi_{\mathbf{k}}}{\partial (\beta \mu)} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} + 1}. \tag{2}$$

In the following, we will restrict ourselves to three-dimensional particles with quadratic dispersion: $E_{\mathbf{k}} = \hbar^2 k^2 / 2m$. The total particle number is given by:

$$N = \sum_{k} n_{\mathbf{k}} = V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{e^{-(E_{\mathbf{k}} - \mu)/k_B T} + 1}, \tag{3}$$

which very similar to the case of bosons, become the density and chemical potential relation:

$$n\lambda_T^3 = f_{3/2}(z), \tag{4}$$

where $\lambda_T = h/(2\pi m k_B T)^{1/2}$ is the thermal wavelength, $z = e^{\beta \mu}$ is the fugacity and a special function

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^{+\infty} \frac{x^{\nu-1}}{e^x/z + 1} dx, \tag{5}$$

whose shape is plotted below. Its asymptotic behavior is

$$f_{3/2}(z) \approx \begin{cases} z - 2^{-2/3} z^2, & z \ll 1, \\ \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left(1 + \frac{\pi^2}{8} (\ln z)^{-2}\right), & z \gg 1. \end{cases} \tag{6}$$
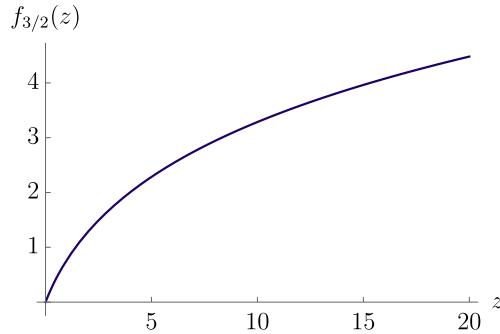

Figure 1: Plot of $f_{3/2}(z)$ defined in Eq. (5).

One should be cautious that in general μ should be solved according to Eq. (4), and is temperature dependent. Therefore, it is not very straightforward what the physical meaning of large or small z limit is.

Let us first focus on the low temperature limit, where $\mu \sim E_F$ and $\beta \mu \gg 1$. Therefore, we are working in the large z limit. Using the large z asymptotic behavior, Equation (4) becomes

$$n \left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} = \frac{4}{3\sqrt{\pi}} \left(\frac{\mu}{k_B T}\right)^{3/2}, \tag{7}$$

which finally reduces to

$$\mu = \frac{\hbar^2 k_F^2}{2m}, \quad k_F = (6\pi^2 n)^{1/3}, \tag{8}$$

which is identical to the Fermi energy E_F.

We now study the high temperature limit. The L.H.S. of Eq. (4) can be as small as possible, in which case the R.H.S. must be $z \ll 1$. This can only be possible if $\mu < 0$. Indeed, use the small z asymptotic behavior, Equation (4) becomes $n\lambda_T^3 = z$, which leads to

$$\left(\frac{T_0}{T}\right)^{3/2} = f_{3/2}(z) \approx z, \tag{9}$$

and

$$\mu = k_B T \ln(n\lambda_T^3) \sim -\frac{3}{2}T \ln T, \tag{10}$$

In Figure 2 we plot the numerical solution of $\mu(T)$, which is consistent with our analysis in both limits.
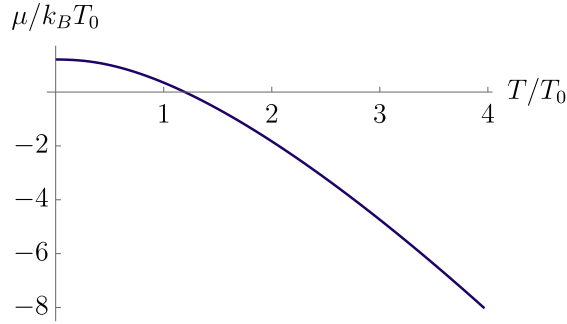

Figure 2: Chemical potential as a function of the temperature. Here $T_0 = n(h^2/2\pi m k_B)^{3/2}$.

Now we are ready to discuss how does Fermi-Dirac distribution become classical. If one naively sets $T \to \infty$ and ignores the temperature dependence of the chemical potential, the result will be wrong:

$$\lim_{T \to +\infty} \frac{1}{e^{(E-\mu)/k_B T} + 1} = \frac{1}{2}. \tag{11}$$

Take the temperature dependence of chemical potential into account:

$$n_F(E) = \frac{1}{e^{E/k_B T}/z + 1} \sim \frac{1}{e^{E/k_B T}T^{3/2}/T_0^{3/2} + 1} \sim T^{-3/2}. \tag{12}$$

The additional $T^{3/2}$ factor due to the chemical potential is vital. Because the denominator is large, the $+1$ term can be neglected and the Fermi-Dirac distribution becomes classical Boltzmann distribution. Physically, when the probability for a state to be occupied is much smaller than one, the Pauli exclusion principle will not be effective and there is no difference between quantum and classical.

2 Relativistic Fermi Gas

Consider a relativistic Fermi gas in three dimensions, whose dispersion is given by

$$E_{\mathbf{k}} = \sqrt{\hbar^2 c^2 k^2 + m^2 c^4} - mc^2. \tag{13}$$

There are three energy scales in the problem: Temperature $k_B T$, Fermi energy E_F, Relativistic energy mc^2. which comes from state counting in the momentum space. No complicated calculation involving density of states is needed. In this case, when we talk about high temperature, it is not clear what this means exactly.

3

We first consider classical and quantum transition. Quantum limit is when $k_B T \ll E_F$. As is discussed in the previous section, in this case, the Fermi-Dirac distribution and Boltzmann distribution are essentially the same.

We then consider non-relativistic and relativistic transition. From the dispersion point of view,

$$E_{\mathbf{k}} \sim \begin{cases} \frac{\hbar^2 k^2}{2m}, & \hbar k \ll mc, \\ \hbar c |k| - mc^2, & \hbar k \gg mc. \end{cases} \tag{14}$$

The particle essentially has a quadratic dispersion In the non-relativistic limit and has a linear dispersion in the relativistic limit. In this way, the relativistic limit is when $k_B T \gg mc^2$ or $E_F \gg mc^2$. For the former case, there are so many particles are excited with linear dispersion that we can effectively treat the particle to have linear dispersion. For the latter case, only particles near E_F are excited, which all have linear dispersion.

We are now ready to draw the $T \sim n$ "phase" diagram for relativistic Fermi gas. Fermi energy is related to the particle density as

$$k_F = (6\pi^2 n)^{1/3}, \tag{15}$$

so that

$$\frac{E_F}{mc^2} = \sqrt{\left(\frac{n}{n_0}\right)^{2/3} + 1} - 1, \quad n_0 = \frac{4\pi}{3}\left(\frac{mc}{h}\right)^3. \tag{16}$$
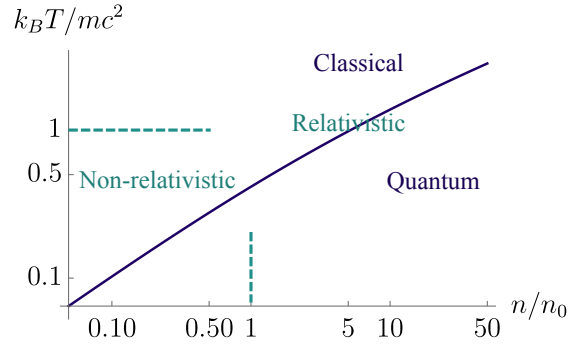

Figure 3: "Phase diagram" of relativistic Fermi gas.

Note that there are actually no phase transitions at all and all lines only indicate crossovers, i.e. a continuous transition even in the thermodynamic limit.

3   Absorption of Fermi Gas

Consider a three-dimensional Fermi gas with quadratic dispersion. There are N such molecules in thermal equilibrium with N impurity sites. Each site can host at most one fermion molecule and the binding energy of the impurity site is $-\varepsilon$. It is not hard to see

$$N_f = V \int \frac{d\mathbf{k}}{e^{\beta(E_{\mathbf{k}} - \mu)} + 1} = \frac{V}{\lambda_T^3} f_{3/2}(z), \tag{17}$$

$$N_b = \frac{N}{e^{-\beta(\varepsilon + \mu)} + 1}, \tag{18}$$

and the chemical potential is determined by

$$N = N_f + N_b = \frac{V}{\lambda_T^3} f_{3/2}(z) + \frac{N}{e^{-\beta(\varepsilon + \mu)} + 1}, \tag{19}$$

4

which transfers to

$$n\lambda_T^3 = (1 + e^{\beta\varepsilon}z)f_{3/2}(z). \tag{20}$$

One should again be cautious about the meaning of "high" or "low" temperature, as there are three energy scales: $k_B T$, E_F and ε in the problem.

Consider $z \ll 1$ limit, to the quadratic order of z:

$$n\lambda_T^3 \approx (1 + e^{\beta\varepsilon}z)(z - 2^{-3/2}z^2) \approx z + (e^{\beta\varepsilon} - 2^{-3/2})z^2, \tag{21}$$

which gives the solution

$$z = \frac{\sqrt{4n\lambda_T^3(e^{\beta\varepsilon} - 2^{-3/2}) + 1} - 1}{2(e^{\beta\varepsilon} - 2^{-3/2})}. \tag{22}$$

Note that according to the solution, $z \ll 1$ if $n\lambda_T^3 \ll 1$ or $e^{\beta\varepsilon} \gg 1$.

- High temperature $n\lambda_T^3 \ll 1$.

$$n_f \approx \frac{z}{\lambda_T^3} \approx \frac{1}{\lambda_T^3}\frac{\sqrt{4n\lambda_T^3(e^{\beta\varepsilon} - 2^{-3/2}) + 1} - 1}{2(e^{\beta\varepsilon} - 2^{-3/2})} \approx n, \tag{23}$$

  which makes sense as at high temperature we expect most particles are not absorbed.

- Low temperature $e^{\beta\varepsilon} \gg 1$.

$$n_f \approx \frac{z}{\lambda_T^3} \approx \frac{1}{\lambda_T^3}\frac{\sqrt{4n\lambda_T^3(e^{\beta\varepsilon} - 2^{-3/2}) + 1} - 1}{2(e^{\beta\varepsilon} - 2^{-3/2})} \approx \sqrt{n\lambda_T^{-3}}e^{-\beta\varepsilon/2}, \tag{24}$$

  which is expected because at exactly the ground state all particles will occupy negative energy impurity states.

It is very important to notice both high and low temperature corresponds to the $z \ll 1$ limit, due to the nontrivial temperature dependence of the chemical potential. The numerical solution of Eq. (20) is given in the figure below, which is consistent with our analysis above.
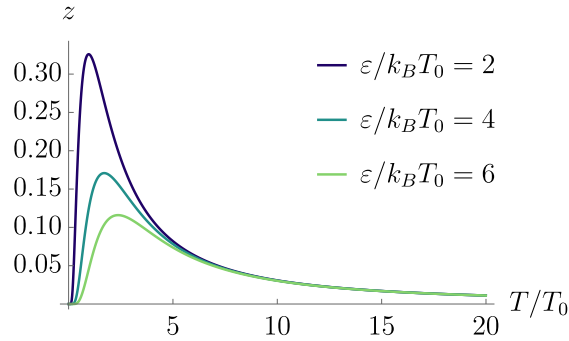

Figure 4: Fugacity z as a function of the temperature. Here $T_0 = n(h^2/2\pi m k_B)^{3/2}$.