

8.08 Statistical Physics II — Spring 2019

Final Review

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The final will be mostly focused on the phase transition, Fermi statistics, and the kinetic theory.

1 Phase Transition

To review Ginzburg-Landau Theory, see Recitation Note 4 and 6. Here we only cover one practice problem.

1.1 Mean-Field Solution of Ising Model

Consider 2D Ising model on square lattice where neighboring spins interact. A horizontal neighboring pair of spins has an interaction energy $-J_x s_i s_{i+x}$. A vertical neighboring pair of spins has an interaction energy $-J_y s_i s_{i+y}$. Use the mean-field theory to compute the phase transition temperature.

The total energy of the system can be written as

$$E = \frac{1}{2} \sum_i -J_x (s_i s_{i+x} + s_i s_{i-x}) - J_y (s_i s_{i+y} + s_i s_{i-y}), \quad (1)$$

where the 1/2 factor is because each link is over-counted twice by summing over all sites. The order parameter is

$$M = \frac{1}{N} \sum_i \langle s_i \rangle. \quad (2)$$

Using the mean-field decomposition $s_i = M + (s_i - M)$ and assuming $(s_i - M)$ is small, to the first order of $(s_i - M)$,

$$E = \frac{1}{2} \sum_i -J_x [M(2s_i + s_{i+x} + s_{i-x}) - 2M^2] - J_y [M(2s_i + s_{i+y} + s_{i-y}) - 2M^2] \quad (3)$$

$$= \sum_i -(J_x + J_y)(2Ms_i - M^2). \quad (4)$$

If you are familiar with the mean-field solution of isotropic Ising model, this already looks like that with $z = 2$ and $J = J_x + J_y$. Directly use the conclusion, we have

$$T_c = \frac{Jz}{k_B} = \frac{2(J_x + J_y)}{k_B}. \quad (5)$$

As a sanity check, when $J_x = J_y = J$, the system is the same as an isotropic Ising model with $z = 4$.

We can also compute T_c from scratch. The key is to notice Eq. (4) is just the energy of N independent spins in an effective magnetic field $B_{\text{eff}} \equiv 2(J_x + J_y)M$. The self-consistency relation is

$$M = \frac{\sum_{s_1=\pm 1} s_1 e^{-\beta B_{\text{eff}} s_1}}{\sum_{s_1=\pm 1} e^{-\beta B_{\text{eff}} s_1}} = \tanh(\beta B_{\text{eff}}) = \tanh[2\beta(J_x + J_y)M]. \quad (6)$$

The critical condition for the nonzero M solution is when the slope of the R.H.S. is the same as that of the L.H.S., which is simply $2\beta_c(J_x + J_y) = 1$. This is identical to Eq. (5).

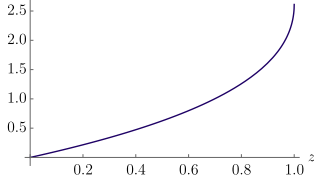
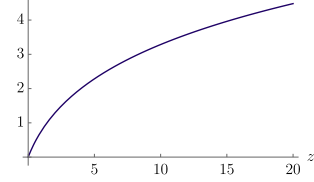
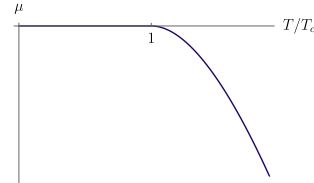
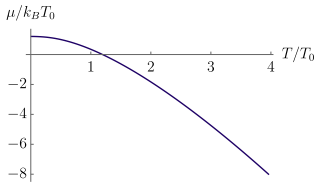
One can also obtain the self-consistency relation by minimizing the free energy, which arguably requires more efforts. See Recitation Note 5 for details.

2 Quantum Statistics

2.1 Basic Properties

Here we summarize basic properties of three-dimensional quadratic dispersed non-interacting particles with different quantum statistics. The Hamiltonian is

$$H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} = \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m}. \quad (7)$$

Statistics	Bose-Einstein	Fermi-Dirac
Grand partition function $\Xi_{\mathbf{k}}$	$[1 - e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}]^{-1}$	$[1 + e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}]^{+1}$
Particle number distribution $n_{\mathbf{k}}$	$\frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} - 1}$	$\frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} + 1}$
$\mu \sim n$ relation	$n\lambda_T^3 = g_{3/2}(z)$	$n\lambda_T^3 = f_{3/2}(z)$
		
$\mu \sim T$ relation		
Low temperature physics	Bose-Einstein condensation	Degeneracy pressure

Here $\lambda_T = h/(2\pi m k_B T)^{1/2}$ is the thermal wavelength, $z = e^{\beta\mu}$ is the fugacity, and $f_{3/2}$ and $g_{3/2}(z)$ are special functions whose plots are shown in the table. The fact that $g_{3/2}(z)$ is bounded leads to the Bose-Einstein condensation below T_c .

It is important to note that μ is temperature dependent and when we take large T limit, we must also take this dependence into account in order to reproduce the classical behavior. For both Bose and Fermi gas, at high temperature $\mu \rightarrow -\infty$ and $z \rightarrow 0$, which leads to $n\lambda_T^3 \ll 1$. Physically, this means the distance between particles is much larger than their thermal wavelength—the characteristic length scale of the particle in the context of statistic mechanics. Therefore, particles do not overlap and quantum statistics is ineffective.

See Recitation Note 8 and 11 for more details on Bose gas and Fermi gas respectively.

2.2 Absorption of Fermi Gas

We review a problem in the pset. For the purpose of the final, you do not need to know the asymptotic expansion of $f_\nu(z)$.

Consider a three-dimensional Fermi gas with quadratic dispersion. There are N such molecules in thermal equilibrium with N impurity sites. Each site can host at most one fermion molecule and the binding energy of the impurity site is $-\varepsilon$. Find the density of free particles in the low and high temperature limits.

For this type of problems, the first step is always to figure out the $\mu \sim n$ (or $z \sim n$) relation and go from there. It is not hard to see

$$N_f = V \int \frac{d\mathbf{k}}{e^{\beta(E_{\mathbf{k}} - \mu)} + 1} = \frac{V}{\lambda_T^3} f_{3/2}(z), \quad (8)$$

$$N_b = \frac{N}{e^{-\beta(\varepsilon + \mu)} + 1}, \quad (9)$$

and the chemical potential is determined by

$$N = N_f + N_b = \frac{V}{\lambda_T^3} f_{3/2}(z) + \frac{N}{e^{-\beta(\varepsilon + \mu)} + 1}, \quad (10)$$

which transfers to

$$n\lambda_T^3 = (1 + e^{\beta\varepsilon} z) f_{3/2}(z). \quad (11)$$

This equation cannot be solved analytically, so we need to analyze its asymptotic behavior. Consider $z \ll 1$ limit, to the quadratic order of z :

$$n\lambda_T^3 \approx (1 + e^{\beta\varepsilon}z)(z - 2^{-3/2}z^2) \approx z + (e^{\beta\varepsilon} - 2^{-3/2})z^2, \quad (12)$$

where we have used the expansion of $f_{3/2}(z)$: $f_{3/2}(z) \approx z - 2^{-2/3}z^2$ for $z \ll 1$. This quadratic equation of z gives the solution

$$z = \frac{\sqrt{4n\lambda_T^3(e^{\beta\varepsilon} - 2^{-3/2}) + 1} - 1}{2(e^{\beta\varepsilon} - 2^{-3/2})}. \quad (13)$$

Note that according to the solution, $z \ll 1$ if $n\lambda_T^3 \ll 1$ or $e^{\beta\varepsilon} \gg 1$.

- High temperature $n\lambda_T^3 \ll 1$.

$$n_f \approx \frac{z}{\lambda_T^3} \approx \frac{1}{\lambda_T^3} \frac{\sqrt{4n\lambda_T^3(e^{\beta\varepsilon} - 2^{-3/2}) + 1} - 1}{2(e^{\beta\varepsilon} - 2^{-3/2})} \approx n, \quad (14)$$

which makes sense as at high temperature we expect most particles are not absorbed.

- Low temperature $e^{\beta\varepsilon} \gg 1$.

$$n_f \approx \frac{z}{\lambda_T^3} \approx \frac{1}{\lambda_T^3} \frac{\sqrt{4n\lambda_T^3(e^{\beta\varepsilon} - 2^{-3/2}) + 1} - 1}{2(e^{\beta\varepsilon} - 2^{-3/2})} \approx \sqrt{n\lambda_T^{-3}} e^{-\beta\varepsilon/2}, \quad (15)$$

which is expected because at exactly the ground state all particles will occupy negative energy impurity states.

It is very important to notice both high and low temperature corresponds to the $z \ll 1$ limit, due to the nontrivial temperature dependence of the chemical potential.

See Recitation Note 11 for more problems related to Fermi gas.

3 Kinetic Theory

We review two problems in the last pset.

3.1 Effusion

A high vacuum chamber develops a crack of area σ , and air from outside leaks in by effusion.

(a) Find the rate the air molecules leaking in through the crack.

(b) The leak is soon stopped, after a small amount of air enter the chamber. After the air in the chamber come to equilibrium, show that its temperature is higher then that of the air outside by a factor $4/3$.

The particle flux is

$$I_N = \sigma n \int_{v_x > 0} d^3 \mathbf{p} v_x f(\mathbf{p}) = \frac{\sigma n}{m} \int_{p_x > 0} d^3 \mathbf{p} p_x f(\mathbf{p}), \quad (16)$$

where $f(\mathbf{p})$ is the normalized Maxwell-Boltzmann distribution:

$$f(\mathbf{p}) = (2\pi m k_B T)^{-3/2} e^{-\beta p^2/2m}, \quad (17)$$

with $p^2 \equiv p_x^2 + p_y^2 + p_z^2$. Evaluate the integral:

$$I_N = \frac{\sigma n}{m} (2\pi m k_B T)^{-1/2} \int_0^{+\infty} dp_x p_x e^{-\beta p_x^2/2m} = \sigma n \sqrt{\frac{m k_B T}{2\pi}} = \sigma n \sqrt{\frac{k_B T}{2\pi m}}. \quad (18)$$

The energy flux is

$$I_E = \sigma n \int_{v_x > 0} d^3 \mathbf{p} v_x \frac{p^2}{2m} f(\mathbf{p}) = \frac{\sigma n}{2m^2} \int_{p_x > 0} d^3 \mathbf{p} p_x p^2 f(\mathbf{p}) = 2k_B T I_N. \quad (19)$$

(You will be given the integral formula, if you are asked to evaluate complicated integrals in the final.)

Therefore, the average energy of escaped molecules is

$$\langle E \rangle = \frac{I_E \Delta t}{I_N \Delta t} = \frac{I_E}{I_N} = 2k_B T. \quad (20)$$

Because of the conservation of energy, the internal energy is also $U = \langle E \rangle = 2k_B T$. On the other hand, the internal energy of classical ideal gas at equilibrium is

$$U = \frac{3}{2} k_B T'. \quad (21)$$

Equate the two internal energies, we have

$$T' = \frac{4}{3} T. \quad (22)$$

3.2 Diffusion and Random Walk

Regard the atomic motion in a gas as a random walk due to collisions. Give an order-of-magnitude estimate of the time it would take an air molecule in a room to travel a distance of 1cm. What about 1m? (You need to estimate the mean-free path first.)

Denote the density of air as n , the scattering length as σ . $\sigma = \pi r^2$, where r is the kinetic radius of the gas molecule. The mean-free path is $\lambda = 1/n\sigma$. This is the distance traveled by the gas molecule in a step of random walk. In order to traverse a distance of L , the particles on average need $N = L^2/\lambda^2$ steps of random walks (because of the standard deviation of random walk). Each step takes time $\tau = \lambda/v$. In this way, the total time needed to travel the distance of L through random walk is

$$T = N\tau = \frac{L^2 \lambda}{\lambda^2 v} = \frac{L^2 \lambda}{v} = \frac{L^2}{\pi v n r^2}. \quad (23)$$

Insert the numbers, you will find that it takes around 10mins or even longer in order for the gas molecule to travel 1m. This means when you suddenly smell the sweet popcorn when your friend brings it out from the bag, the smelling is very likely NOT due to the diffusion, but is due to the air flow induced by the pressure difference.

3.3 Sound Wave

You will not be asked to derive the wave equation of sound waves in the final, which means no Euler equation nor Navier-Stokes equations (yay!). However, you should know what the wave equation is and how to derive the sound velocity from it. The second part of problem 4 in pset 11, where the damping coefficient is derived from the wave equation, is a good reference.