8.08 Statistical Physics II — Spring 2019
Recitation Note 4

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Contents

It was Landau who first classified phases of matter according to orders. Orders originate from broken symmetry and are characterized by "order parameters". This framework, usually called "Landau's paradigm", can capture a wide range of phases in classical and quantum systems. Here are some examples of ordered phases:

- Crystals. Continuous translation symmetry is broken down to discrete translation symmetry.
- Magnets. Time-reversal and spin rotation symmetry is broken.
- Nematic liquid crystals. Rotational symmetry is broken (but continuous translation symmetry is still preserved).
- Superfluid helium. Global $U(1)$ symmetry of the phase of the helium wavefunction is broken.

The last example is more subtle as the broken symmetry is intrinsically quantum mechanical.

In Landau's paradigm, the vapor and the liquid of water are regarded as the same phase because they are not only connected in the phase diagram (beyond the critical point), but also share the same order parameter.

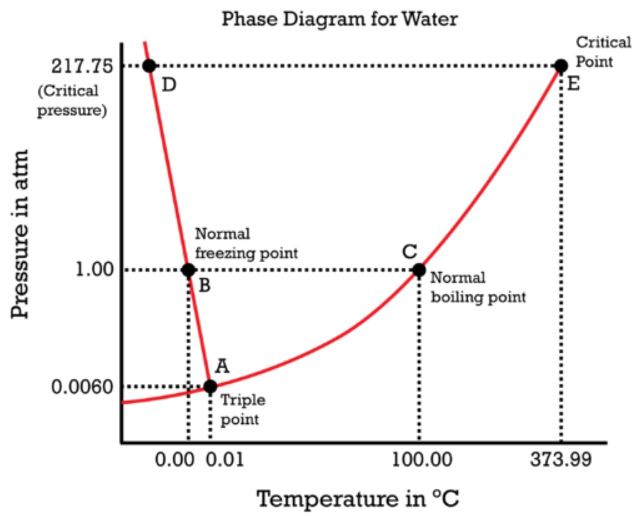

Figure 1: Phase diagram of water.

Phases such as the (fractional) quantum Hall states cannot be fit into the Landau paradigm, as there does not exist any local order parameters that can characterize the phase. Concepts such as the "topological order" is coined to characterize such phases.

1 Phenomenological: Ginzburg-Landau Theory

We study the spin ferromagnet-paramagnet phase transition. From the experiment, we find when the temperature is below some critical temperature T_c, the magnetization starts to grow from zero. The order parameter is thus the magnetization:

$$M = \frac{1}{N}\sum_{i=1}^{N}\langle S_i\rangle, \tag{1}$$

where N is the total spin number and $\langle S_i\rangle$ is the expectation of the i-th spin.

In the following, we construct a phenomenological theory to describe the thermodynamic property of the system near the phase transition. The goal of the phenomenological theory is to describe the empirical relationship between different phenomena. It is not directly derived from the first principles, for example, by solving the Schödinger equation. Instead, the phenomenological theory is only required to be consistent with the first principles. The parameters in the phenomenological theory can either be fitted from the experiment data, or in some cases can be derived from the microscopic theory.

Note that S_i in Eq. (1) can be discrete or continuous depending on the level of approximation. Since right now we are working out a phenomenological theory, this detail does not matter.

1.1 Zero Magnetic Field

Near the critical point, M is small and the free energy can be Taylor expanded as a series of M. We first assume there is no external magnetic field, such that the energy of the system is invariant under spin flip operation $S_i \to -S_i$ for all spins i. In this way, the free energy should also be invariant under $M \to -M$. Therefore, odd order terms cannot appear in the expansion. Also the free energy must be extensive with the spin number N. This leads to

$$F(T,M) = NF_0(T) + \frac{1}{2}Na(T)M^2 + \frac{1}{4}Nb(T)M^4. \tag{2}$$

For convenience, in the following we will work with the free energy density

$$f(T,M) = \frac{F}{N} = F_0(T) + \frac{1}{2}a(T)M^2 + \frac{1}{4}b(T)M^4. \tag{3}$$

The magnetization is chosen such that the free energy is minimized. This requires $b(T) > 0$ for all T, otherwise the minimum is always at infinity. The solution to

$$\frac{\partial f}{\partial M} = a(T)M + b(T)M^3 = 0, \tag{4}$$

is

$$M = 0,\ \pm\sqrt{-\frac{a(T)}{b(T)}}. \tag{5}$$

Since M must be real, the second solution exists only when $a(T)/b(T) < 0$, i.e. $a(T) < 0$. Therefore, $a(T)$ must changes sign when the temperature goes across the critical temperature T_c.

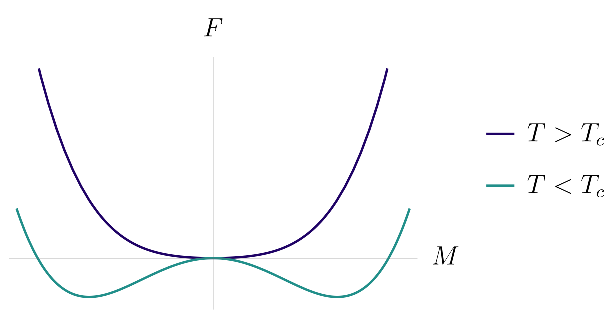

Figure 2: Shape of the free energy as a function of the order parameter below the critical temperature. Here the magnetic field is nonzero so that the degeneracy is broken.

The above analysis allow us to expand the parameters $a(T)$ and $b(T)$ as functions of temperature T around T_c to the lowest nonzero order:

$$F_0(T) = F_0, \tag{6}$$
$$a(T) = a_1(T - T_c), \tag{7}$$
$$b(T) = b_0, \tag{8}$$

where $a_1, b_0 > 0$.

To this end, we can already compute many thermodynamic properties of the system:

- Magnetization.

$$M = \begin{cases} 0, & T > T_c, \\ \pm\sqrt{\frac{a_1(T_c-T)}{b_0}} = \pm\sqrt{\frac{a_1 T_c}{b_0}t}, & T \le T_c, \end{cases} \tag{9}$$

where

$$t \equiv 1 - \frac{T}{T_c}. \tag{10}$$

Note that when $T < T_c$ there are two solutions of M with equal energy but opposite sign. In reality, at the critical temperature, the system will randomly pick one of the solution due to the thermal fluctuation. This is called "spontaneous symmetry breaking"

- Free energy density.

$$f(T) = F_0 + \frac{1}{2}a_1(T-T_c)M^2 + \frac{1}{4}b_0 M^4 \tag{11}$$

$$= \begin{cases} F_0, & T > T_c, \\ F_0 - \frac{a_1^2(T-T_c)^2}{4b_0}, & T \le T_c. \end{cases} \tag{12}$$

Note that the free energy is continuous around T_c.

- Entropy density.

$$s = -\frac{\partial f}{\partial T} = \begin{cases} 0, & T > T_c, \\ \frac{a_1^2(T-T_c)}{2b_0}, & T \le T_c. \end{cases} \tag{13}$$

Note that the entropy is also continuous around T_c. This means there is no latent heat during the phase transition.

- Heat capacity per spin.

$$c_V = T\frac{\partial s}{\partial T} = \begin{cases} 0, & T > T_c, \\ \frac{a_1^2 T}{2b_0}, & T \le T_c, \end{cases} \tag{14}$$

which has a discontinuous jump across T_c. Because the first order derivative of the free energy is continuous while the second order derivative is not, this phase transition is classified as the "second order phase transition".

1.2 Nonzero Magnetic Field

We now add small magnetic field to the system. Magnetic field breaks the spin flip symmetry. To the lowest order, this will introduce a term linear in M to the free energy density: $c(B,T)M$. Expand $c(B,T)$ to the lowest order in both B and T, we have

$$f(T,B) = F_0 + c_0 BM + \frac{1}{2}a_1(T-T_c)M^2 + \frac{1}{4}b_0 M^4. \tag{15}$$

The linear term breaks the symmetry between the two degenerate state when $B = 0$.

When $T > T_c$, it suffices to keep only the quadratic term. Minimizing the free energy gives

$$M = \frac{c_0 B}{a_1(T_c - T)}. \tag{16}$$

When $T < T_c$, we apply perturbation theory to the original zero-field solution Eq. (9). Denote $M_0 = \sqrt{a_1 T_c t/b_0} > 0$ such that $M = \pm M_0 + \delta M(B)$. The sign of M_0 depends on the sign of $c_0 B$.

Assume δM is small, we can expand $M^n = (M_0 + \delta M)^n \approx M_0^n + nM_0^{n-1}\delta M$. In this way, the equation $\partial f/\partial M = 0$ becomes

$$c_0 B + a_1(T-T_c)(M_0 + \delta M) + b_0(M_0 + 3M_0^2\delta M) = 0, \tag{17}$$

whose solution is

$$\delta M = \frac{-c_0 B}{a_1(T - T_c) + 3b_0 M_0^2} = \frac{c_0 B}{2a_1(T - T_c)}. \tag{18}$$

The above result allows us to compute the magnetic susceptibility of the system:

$$\chi = \frac{\partial M}{\partial B}\Big|_{B\to 0} = \begin{cases} \frac{c_0}{a_1(T_c - T)} = \frac{c_0}{a_1 T_c} t^{-1}, & T > T_c, \\ \frac{c_0}{2a_1 T_c}(-t)^{-1}, & T \leq T_c. \end{cases} \tag{19}$$

Note that the magnetic susceptibility diverges at the critical temperature. Also, the $1/T$ dependence at high temperature is consistent with the Curie's law of paramagnetism.

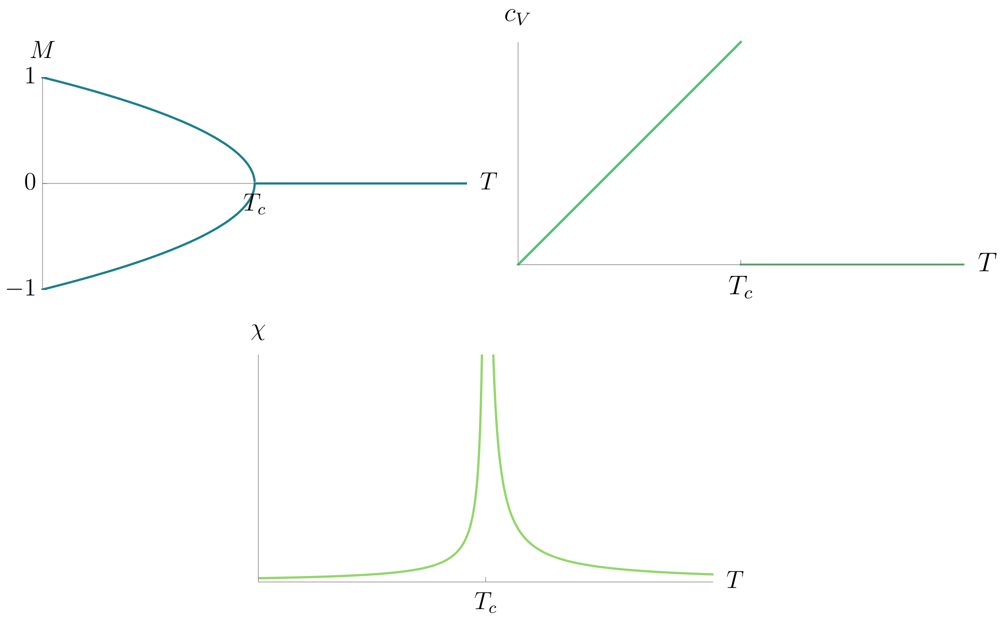

Figure 3: Magnetization, heat capacity and magnetic susceptibility of the Ginzburg-Landau theory as functions of temperature.

Special attention should be paid to the phase transition controlled by the magnetic field when $T < T_c$. Without loss of generality assume $c_0 > 0$. When $B > 0$, the solution to the free energy is $M = -M_0 + \delta M$. When $B < 0$, the solution suddenly jumps to $M = M_0 + \delta M$. Because the first order derivative of the free energy is discontinuous ($M \propto \partial F/\partial B$), this phase transition is classified as the "first order phase transition".

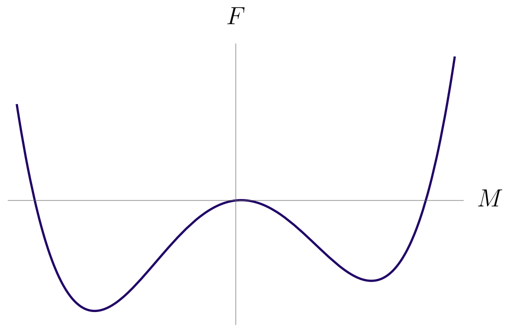

Figure 4: Shape of the free energy as a function of the order parameter below and above the critical temperature. There is no magnetic field so that there are two degenerate spin states.

This type of analysis, by expanding the free energy as a function of the order parameter, is called the Ginzburg-Landau theory. Here are several remarks about the theory:

- When the interaction is present, the free energy is normally not an analytic function at the phase transition point. For example, there may be a logarithmic dependence. In these cases, one cannot Taylor expand the free energy as what we have done above;
- Ginzburg-Landau theory can also capture the vapor-liquid phase transition of the water, because nothing can prevent us from expanding the free energy as a function of density difference of the vapor and the liquid. However, the density difference in this case is not an order parameter.
- Landau's paradigm requires symmetry breaking to be associated with second order phase transition. Nothing to much can be said about the first order phase transition. There is no symmetry breaking involved in the above example of magnetic field driven first order phase transition.