8.231 Physics of Solids I — Fall 2017
Problem Set 8

Posted: Tuesday, Nov 14, 2017
***Due:* Tuesday, Nov 21, 2017**

Readings (Optional)

- L. N. Cooper, "Bound Electron Pairs in a Degenerate Fermi Gas", *Phys. Rev.* **104**, 1189 (1956).
- J. Bardeen, L. N. Cooper, and J. R. Schrieffer, "Microscopic Theory of Superconductivity", *Phys. Rev.* **106**, 162 (1957).

Problem 1

Bose-Einstein Condensation

Consider a large ensemble of N free bosons with dispersion $\varepsilon_{\mathbf{k}} = \hbar^2 k^2/(2m)$. At thermal equilibrium, their distribution satisfies the Bose-Einstein distribution function:

$$n(\mathbf{k}) = \frac{1}{e^{(\varepsilon_{\mathbf{k}}-\mu)/(k_B T)} - 1}. \tag{1}$$

(a) Write down the equation for total particle number conservation. Note here the total boson number N is fixed and the chemical potential μ is varying and is a function of temperature T.

(b) The chemical potential $\mu(T) \leq 0$ in order to make $n(\mathbf{k}) \geq 0$. Evaluate the temperature T_c when $\mu(T_c) = 0$ for three dimensional bosons. You will find this integral useful:

$$\int_0^{+\infty} \frac{\sqrt{z}}{e^z - 1} = \frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right), \tag{2}$$

where $\zeta(z)$ is the Riemann zeta function.

(c) What happens when $T < T_c$? What happens when $T = 0$?

(d) Comment on the Bose-Einstein condensation at one and two dimensions.

Problem 2

Pairing Instability of the Fermi Surface

In this problem, we will repeat the calculation in the seminal paper by Leon N. Cooper back in 1956: *Bound Electron Pairs in a Degenerate Fermi Gas.* This paper lies the theoretical foundation of BCS superconductivity theory.

Part I: Electrons in the Vacuum First consider two electrons in the *vacuum*. The Schrödinger equation of the system is given by

$$\left(-\frac{\hbar^2\nabla_1^2}{2m} - \frac{\hbar^2\nabla_2^2}{2m} + V(\mathbf{r}_2 - \mathbf{r}_1)\right)\Psi(\mathbf{r}_1, \mathbf{r}_2) = E\Psi(\mathbf{r}_1, \mathbf{r}_2), \tag{3}$$

where $\mathbf{r}_i$ is the position of i-th electron. m is the electron mass.

(a) First rewrite the Schrödinger equation in the center-of-mass frame and consider solutions with zero center-of-mass momentum. Then expand the wavefunction with plane waves. Show that the Schrödinger reduces to

$$2\varepsilon_{\mathbf{k}}\Psi(\mathbf{k}) + \int d\mathbf{k}' V(\mathbf{k} - \mathbf{k}')\Psi(\mathbf{k}') = E\Psi(\mathbf{k}). \tag{4}$$

Identify $\varepsilon_{\mathbf{k}}$ by yourself.

(b) Suppose the interaction potential is $V(\mathbf{r}) = gV\delta(\mathbf{r})$, where $\delta(\mathbf{r})$ is the Dirac delta function. Show that the Schrödinger equation only has negative energy solutions $E < 0$. And these solutions exist when $g < 0$, i.e. there are bound states only when the potential is attractive.

(Hint: You need to take a cutoff $k_{\max}$ in order for the momentum integral to converge due to the special form of the delta potential.)

Part II: Electrons in the Metal Now consider a two-electron excitation in the metal, i.e., two electrons in the metal with momenta above the Fermi momentum. Therefore, the momentum summation in the Schrödinger equation is now restricted to $k > k_F$ because states with $k < k_F$ are all occupied by other electrons in the metal.

(a) Assume $k_{\max} \gtrsim k_F$ so that the density of states is nearly constant $D(E_F)$ throughout the integrand in Eq. (4). Solve the Schrödinger equation to obtain $g(E)$ and show there always exist $E < 2E_F$ solutions when $g < 0$ however small $|g|$ is.

(b) Denote the binding energy as $\varepsilon_B \equiv 2E_F - E$ and $E_{\max} - E_F \equiv \hbar\omega_D$. Estimate $\varepsilon_B(g)$. You may assume $\varepsilon_B(g)$ is very small.

In a real superconductor, the microscopic origin of the attractive potential is the electron-phonon interaction. Thus the energy cutoff ω_D is the Debye frequency.

This calculation shows that the total energy of electrons can be lowered if they form the two-electron bound states instead of forming the Fermi sea, given arbitrarily small attraction. This is called the pair instability of the Fermi surface.