8.231 Physics of Solids I — Fall 2017
Problem Set 9

Posted: Tuesday, Nov 28, 2017
***Due:* Tuesday, Dec 5, 2017**

Readings (Optional)

- R. B. Laughlin, "Quantized Hall conductivity in two dimensions", *Phys. Rev. B* **23**, 5632 (1981).
- B. I. Halperin, "Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential", *Phys. Rev. B* **25**, 2185 (1982).

Note: Bonus problems are more challenging. It is fine you do not solve them. By solving them you can earn extra credits.

Problem 1

Berry Phase, Berry Curvature and Chern Number

Consider a Hamiltonian depending on some parameters $\mathbf{R} = (R_1, R_2, \ldots)$: $H = H(\mathbf{R})$ and the parameters in turn depends on the time $\mathbf{R} = \mathbf{R}(t)$. Denote the instantaneous energy eigenstates as

$$H(\mathbf{R})\,|n(\mathbf{R})\rangle = E_n(\mathbf{R})\,|n(\mathbf{R})\rangle\,. \tag{1}$$

Suppose the initial state at $t = 0$ is an instantaneous energy eigenstate:

$$H(\mathbf{R}(0))\,|n(\mathbf{R}(0))\rangle = E_n(\mathbf{R}(0))\,|n(\mathbf{R}(0))\rangle\,, \tag{2}$$

and $\mathbf{R}(t)$ varies slowly with time. According to the adiabatic theorem, at every instant of time, $|n(\mathbf{R}(0))\rangle$ stay as instantaneous energy eigenstate of $H = H(\mathbf{R}(t))$.

(a) Although the state stays as the instantaneous energy eigenstate, its phase factor is yet to be solved. Denote $|\psi_n(t)\rangle = e^{-i\theta(t)}\,|n(\mathbf{R}(t))\rangle$. Show that

$$\theta(t) = \frac{1}{\hbar}\int_0^t E_n(\mathbf{R}(t'))dt' - i\int_0^t \left\langle n(\mathbf{R}(t'))\,\middle|\,\frac{d}{dt'}n(\mathbf{R}(t'))\right\rangle dt'. \tag{3}$$

The first term is called the dynamical phase, which exists even when $\mathbf{R}(t) \equiv \mathbf{R}_0$ is a constant. The second term is called the Berry phase, which will be of our main interest in this problem.

(Hint: Use time-dependent Schrödinger equation.)

(b) Suppose $\mathbf{R}(t)$ moves along a closed path $\mathcal{C}$. Show that the Berry phase can be expressed as a line integral along the closed path:

$$\gamma_n = \oint_{\mathcal{C}} d\mathbf{R} \cdot \mathbf{A}_n(\mathbf{R}). \tag{4}$$

$\mathbf{A}_n(\mathbf{R})$ is called the Berry vector potential or Berry connection. Identify $\mathbf{A}_n(\mathbf{R})$.

As suggested by Eq. (4), Berry phase is independent of how fast the closed path is traversed (as long as adiabatic theorem is satisfied), and is determined solely by the closed path $\mathcal{C}$. Thus it is often called the "geometric phase".

(c) Show that Berry phase is real. Thus it is not "Berry decay".

(Hint: Use the fact $\langle n(\mathbf{R})|n(\mathbf{R})\rangle = 1.$)

(d) Suppose $\mathbf{R} = (R_1, R_2, R_3)$ is of three dimensions. Since $\mathcal{C}$ is a closed path, we can use Stokes' theorem to simplify Eq. (4). Show that Berry phase can also be expressed as

$$\gamma_n = \iint_{\mathcal{C}} d\mathbf{S} \cdot \mathbf{\Omega}_n(\mathbf{R}). \tag{5}$$

$\mathbf{\Omega_n}(\mathbf{R})$ is called the Berry curvature:

$$\Omega_{n,i}(\mathbf{R}) = -i\varepsilon_{ijk} \left\langle \frac{\partial}{\partial R_j} n(\mathbf{R}) \middle| \frac{\partial}{\partial R_k} n(\mathbf{R}) \right\rangle, \tag{6}$$

where we have used Einstein summation convention and ε_{ijk} is the Levi-Civita symbol. Berry curvature can be thought as the magnetic field in the parameter space. Compared with the Berry phase, it is a local quantity that reflects the geometry of the parameter space.

(e) As a concrete example, consider a generic two level system $H = \mathbf{d}(\mathbf{R}) \cdot \boldsymbol{\sigma}$, where the parameterization is on a sphere S^2

$$\mathbf{d}(\mathbf{R}) = |d|(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{7}$$

For the lower energy eigenstate $|-(\mathbf{R})\rangle$, show the Berry curvature

$$\Omega_{-,\theta\varphi} = \frac{\partial A_{-,\varphi}}{\partial\theta} - \frac{\partial A_{-,\theta}}{\partial\varphi} = -\frac{\sin\theta}{2}. \tag{8}$$

(f) The Chern number is the integral of the Berry curvature over the parameterization space. In the previous example, it is computed by

$$C_{\pm} = \frac{1}{2\pi} \int_{S^2} d\theta d\varphi \Omega_{\pm,\theta\varphi}. \tag{9}$$

Compute C_{-}.

(g) (Bonus) According to Stokes' theorem, the integral of the Berry curvature over the parameter space can also be evaluated as the line integral of the Berry vector potential over the boundary of the parameter space. However, since sphere has no boundary, we should expect the integral to be zero. This is not consistent with your calculation in **(f)**. How to you reconcile this paradox?

(Hint: Berry vector potential is gauge dependent.)