8.231 Physics of Solids I — Fall 2017
Recitation 5

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1 Electrons in Magnetic Field

1.1 Classical Mechanics

The Hamiltonian of the electron is

$$H = \frac{1}{2m}\mathbf{\Pi}^2 = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2, \tag{1}$$

where m is the electron mass. $\mathbf{\Pi}$ is called the mechanical/kinetic momentum. $\mathbf{p}$ is called the canonical momentum, which is not a conserved quantity in the presence of the magnetic field. $\mathbf{A}$ is the vector potential that satisfies $\nabla \times \mathbf{A} = \mathbf{B}$.

The Hamiltonian equations of motion is

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right), \tag{2}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} = \frac{e}{mc}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \cdot \nabla \mathbf{A}. \tag{3}$$

Straightforward calculation gives

$$\ddot{\mathbf{x}} = \frac{e}{mc}\left(\mathbf{v} \cdot \nabla \mathbf{A} - (\mathbf{v} \cdot \nabla)\mathbf{A}\right) = \frac{e}{mc}(\mathbf{v} \times \mathbf{B}), \tag{4}$$

which is just the Lorentz force without the electric field. The solution is a cyclotron motion with radius r and angular frequency ω being

$$r = \frac{mv}{eB}, \quad \omega = \frac{eB}{mc}. \tag{5}$$

r is called the "cyclotron radius" and ω is called the "cyclotron frequency".

1.2 Quantum Mechanics

1.2.1 Energy Spectrum

In quantum mechanics, we impose quantize the canonical commutation relation $[x_i, p_j] = ih\delta_{ij}$. It is important to notice that the mechanical momenta do not commute with each other:

$$[\Pi_x, \Pi_y] = \frac{ie\hbar}{c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \frac{ie\hbar}{c}B_z. \tag{6}$$

From this we can define the upper and lower operators:

$$a = \sqrt{\frac{c}{2e\hbar B}}\left(\Pi_x + i\Pi_y\right), \quad a^\dagger = \sqrt{\frac{c}{2e\hbar B}}\left(\Pi_x - i\Pi_y\right). \tag{7}$$

It is straightforward to prove the commutation relation $[a, a^\dagger] = 1$. Also it is not hard to observe the Hamiltonian Eq. (1) can be rewritten as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{8}$$

where ω is the cyclotron frequency. This is exactly the same Hamiltonian as the harmonic oscillator. Denote the eigenstate of $a^\dagger a$ as $|N\rangle$. We have the energy spectrum

$$H|N\rangle = \hbar\omega\left(N + \frac{1}{2}\right)|N\rangle. \tag{9}$$

These energy eigenstates are called Landau levels. N is the Landau level index. The effects of upper and lower operators are

$$a^\dagger|N\rangle = \sqrt{N+1}\,|N+1\rangle, \quad a|N\rangle = \sqrt{N}\,|N-1\rangle. \tag{10}$$

Semi-classical Consideration It is also possible to obtain the spectrum Eq. (9) through a semi-classical approach. According to Sommerfield quantization, for a given orbital of the electron cyclotron motion, we have

$$\begin{aligned}\left(N + \frac{1}{2}\right)h &= \oint \mathbf{p}\cdot d\mathbf{r} && (11)\\ &= \oint\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right)\cdot d\mathbf{r} && (12)\\ &= 2\pi mvr - \frac{e}{c}B\pi r^2 && (13)\\ &= 2\pi mvr - \pi m\omega r^2. && (14)\end{aligned}$$

Thus

$$E = \frac{1}{2}mv^2 = \left(N + \frac{1}{2}\right)\hbar\omega. \tag{15}$$

The fact that even the $\hbar\omega/2$ zero point energy is correctly considered is just a coincidence.

Choices of Vector Potential To this end, we have not specified the form of the vector potential $\mathbf{A}$. In fact, the choice of the vector potentials is not unique, as any gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\varphi$ will not effect the magnetic field. There are two common types of the vector potential:

$$\mathbf{A} = (A_x, A_y) = \begin{cases} B(-y, x)/2, & \text{symmetric gauge,} \\ B(-y, 0), & \multirow{2}{*}{Landau gauge.} \\ B(0, x), & \end{cases} \tag{16}$$

Symmetric gauge preserves the rotational symmetry and the Landau gauge preserves the translational symmetry. It is impossible to choose a gauge that preservers both symmetries.

The choice of the gauge will not effect the physical observables, which are by definition gauge invariant. A wise choice of the gauge will make life easier when solving particular problems. In the following, we will compute the wavefunction under both gauges.

1.3 Symmetric Gauge

Under symmetry gauge, the lower operator in Eq. (7) becomes

$$a = \sqrt{\frac{c}{2e\hbar B}} \left[-\hbar i \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - \frac{eB}{2c} (-y + ix) \right] = -i \sqrt{\frac{\hbar c}{2eB}} \left(2 \frac{\partial}{\partial \bar{z}} - \frac{eB}{2\hbar c} z \right), \tag{17}$$

where

$$\begin{aligned} z &\equiv x + iy, \quad \partial \equiv \frac{\partial}{\partial z} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \bar{z} &\equiv x - iy, \quad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \end{aligned} \tag{18}$$

The ground state is determined by $a |0\rangle = 0$. Denote the wavefunction of the ground state as $\langle z, \bar{z}|0\rangle \equiv \psi_0(z, \bar{z})$. This gives the differential equation

$$\left(2 \frac{\partial}{\partial \bar{z}} - \frac{eB}{2\hbar c} z \right) \psi_0(z, \bar{z}) = 0. \tag{19}$$

The solution is

$$\psi_0(z, \bar{z}) = f(z) \exp \left(-\frac{eB\bar{z}z}{4\hbar c} \right) = f(z) \exp \left(-\frac{\bar{z}z}{4l^2} \right), \tag{20}$$

where $f(z)$ is any continuous function is z so that $\psi_0(z, \bar{z})$ does not blow up at $z = 0$ and $z = \infty$. $l \equiv \sqrt{\hbar c/(eB)}$ is called the "magnetic length".

Ground State Degeneracy The arbitrariness of the $f(z)$ immediately suggests there is huge degeneracy for at least ground state. It is convenient to choose $f(z)$ to be the n-th polynomial of z:

$$\psi_{0,n}(z, \bar{z}) = N_n z^n \exp \left(-\frac{\bar{z}z}{4l^2} \right), \tag{21}$$

where N_n is the normalization factor. n needs to be an non-negative integer to ensure the boundary condition at $r = 0$ and the single-valuedness. In fact, n is also the eigenvalue of the *canonical* angular momentum:

$$L_z = xp_y - yp_x = \hbar(z\partial - \bar{z}\bar{\partial}), \tag{22}$$

and $L_z \psi_{0,n} = n\hbar\psi_{0,n}$. This is a reflection of the rotational invariance of the wavefunction. Note that L_z is not a gauge invariant quantity, and is thus not physical. The physical angular momentum is the *mechanical* angular momentum:

$$\mathcal{L}_z = x\Pi_y - y\Pi_z = L_z - \frac{eB}{2c} r^2, \tag{23}$$

where $r^2 = x^2 + y^2 = \bar{z}z$. It is not hard to prove the expectation value of this mechanical angular momentum for the ground state: $\langle 0, n|\mathcal{L}_z|0, n\rangle = -\hbar$, which is indenpendent of n.

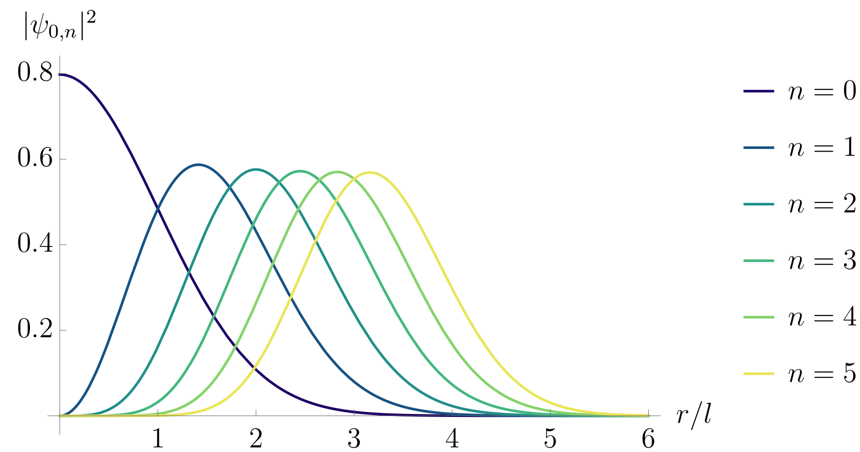

Figure 1: The radial distribution of $|\psi_{0,n}|^2$ for $n = 0, 1, 2, 3, 4, 5$.

The radial probability distribution of $|\psi_{0,n}|^2$ is shown in Figure 1. In fact, the probability is peaked at $r_{\text{max}}^2 = (2n+1)l^2$. This speculation is helpful to count the degeneracy of the state. Suppose the system is a a disk with radius R. Since $\psi_{0,n}$ peaks at $r_{\text{max}}^2 = (2n+1)l^2$, we should have $r_{\text{max}}^2 < R^2$[1]. This means

$$n \le \frac{R^2}{2l^2} = \frac{e\Phi}{hc}, \tag{24}$$

where $\Phi = \pi R^2 B$ is the magnetic flux of the system, and hc/e is the flux quantum. Each state will occupy a flux quantum.

Excited States The excited states can be obtained by acting $a^\dagger$ to the ground state:

$$\psi_{1,n} = a^\dagger \psi_{0,n} = -i\frac{1}{\sqrt{2}l}\left(2n - \frac{\bar{z}z}{l^2}\right) N_n z^{n-1} \exp\left(-\frac{\bar{z}z}{4l^2}\right). \tag{25}$$

It is not hard to show that $L_z\psi_{1,n} = (n-1)h\psi_{1,n}$ and $\langle 1, n|\mathcal{L}_z|1, n\rangle = -3\hbar$, also indenpendent of n. Similar degeneracy counting can be carried out and the higher Landau levels have the same degeneracy $D = e\Phi/(hc)$.

1.4 Landau Gauge

Under Landau gauge (say $\mathbf{A} = B(-y, 0)$), the lower operator in Eq. (7) becomes

$$a = \sqrt{\frac{\hbar c}{2eB}}\left[-i\frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y} + \frac{y}{l^2}\right)\right]. \tag{26}$$

Note that there is no x dependence, k_x is a conserved quantity and we can choose the ground state wavefunction to be $\psi_{0,k_x}(x, y) = e^{ik_x x}\psi_0(y)$. In this way,

$$a = \sqrt{\frac{\hbar c}{2eB}}\left[k_x + \left(\frac{\partial}{\partial y} + \frac{y}{l^2}\right)\right]. \tag{27}$$

The ground state wavefunction $a\,|0\rangle = 0$ becomes

$$\left[k_x + \left(\frac{\partial}{\partial y} + \frac{y}{l^2}\right)\right]\psi_0(x) = 0, \tag{28}$$

[1]Of course, this counting is not exact. Here we are applying hard wall boundary condition, which requires the wavefunction to vanish at $r = R$. However, when we obtain Eq. (20), we actually only require the solution to vanish at infinity so that the wavefunction is normalizable. As long as R is large enough, we should expect this counting to be accurate enough.

which can be solved as

$$\psi_{0,k_x}(x,y) = N_{k_x} \exp\left[ik_x x - \frac{1}{2l^2}\left(y + k_x l^2\right)^2\right], \tag{29}$$

where N_{k_x} is the normalization factor. The probability distribution of $|\psi_{0,k_x}|^2$ is obviously peaked at $y = -k_x l^2$.

We can count the degeneracy similar as what we have done under symmetric gauge. Now suppose the system is a rectangle with size $L_x \times L_y$. $-k_x l^2 \in [0, L_y]$ so that the peak of the wavefunction is within the system. On the other hand, the possible k_y states of plane wave are specified by $k_x = 2\pi n/L_x$. Combine these two, we get

$$n \leq \frac{L_x L_y}{2\pi l^2} = \frac{e\Phi}{hc}, \tag{30}$$

which is exactly the same as Eq. (24).

The excited states can be obtained by acting $a^\dagger$ to the ground state. They have the same degeneracy as the ground state.

2 Integer Quantum Hall Effect

In recitation 1, we introduced the integer quantum Hall effect. Under strong magnetic field, the Hall conductivity forms several plateaus, where the Hall conductivity is quantized

$$\sigma_{xy} = \nu \frac{e^2}{h}. \tag{31}$$

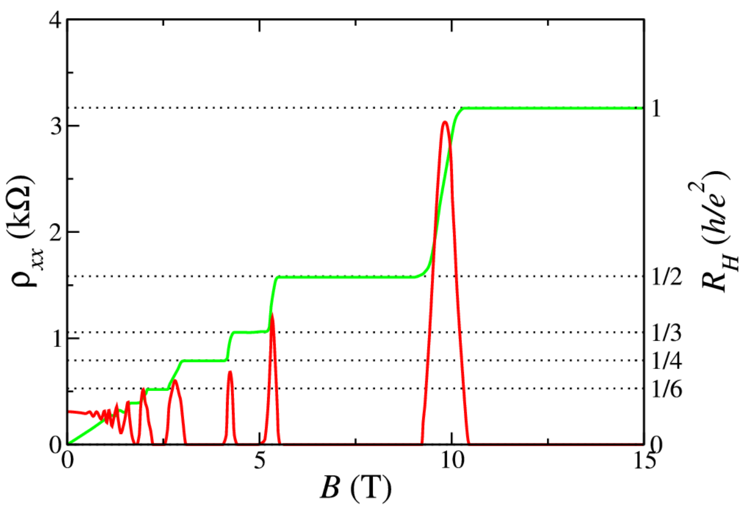

Figure 2: Typical magnetic field dependence of ρ_{xx} (red) and R_H (green) in the strong field limit.

We will try to (partially) understand this phenomenon with the Landau level we solved. The argument is due to Laughlin, and is thus called "Laughlin gauge argument"[1, 2].

2.1 Problem Setup

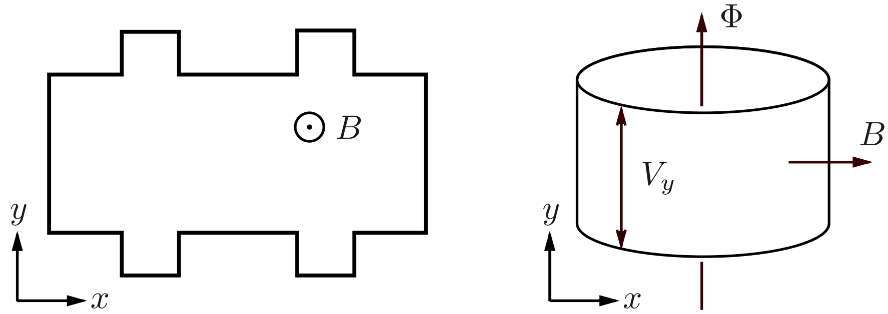

Figure 3: Geometry of the Hall bar (left) and the cylinder geometry in Laugh;in's gauge argument (right).

The geometry of the Hall bar where we measure the Hall effect experimentally is shown in Figure 3. Instead of imposing an external electric field on x direction, we make the system periodic along x direction and thread magnetic flux Φ through the cylinder. When the flux is varied slowly, we should expect an induced electric field along x direction as if there is an external electric field. The full Hamiltonian of the system is then

$$H = \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right) - eE_y y \tag{32}$$

$$= \frac{1}{2m}\left[\left(\hbar k_x + \frac{eBy}{c} - \frac{eA_0}{c}\right)^2 + p_y^2\right] - eE_y y \tag{33}$$

$$= \frac{1}{2m}\left[\left(\hbar k_x + \frac{eBy}{c} - \frac{e\Phi}{cL_x} - \frac{mc}{eB}eE_y\right)^2 + p_y^2\right] + \frac{\hbar c k_x E_y}{B} + \frac{mc^2 E_y^2}{2B^2}. \tag{34}$$

Here A_0 is the vector potential accounting for the threaded flux: $\Phi = A_0 L_x$ according to the Aharonov-Bohm effect. In the last equation, we complete the square to absorb the electric field along y direction into the mechanical momentum. An important observation is that

$$\frac{\partial H}{\partial \Phi} = -\frac{e}{mcL_x}\left(\hbar k_x + \frac{eBy}{c} - \frac{e\Phi}{cL_x} - \frac{mc}{eB}eE\right) = \frac{-e\Pi_x}{mcL_x} = \frac{j_x}{c}, \tag{35}$$

which says change the flux induces a current in the y direction. This is just the Faraday's law of induction.

Although the system is very complicated, we end up getting a Hamiltonian exactly in the same form of the Landau level under Landau gauge. Then we can immediately write down the ground state wavefunction

$$\psi_{0,k_x,\Phi}(x,y) = N_{k_x,\Phi} \exp\left[ik_x x - \frac{1}{2l^2}\left(y + k_x l^2 - \frac{\Phi}{BL_x} - m\left(\frac{c}{eB}\right)^2 eE\right)^2\right]. \tag{36}$$

Excited states can be obtained by applying $a^\dagger$ operators.

The energy for the N-th Landau level is then

$$E_N = \hbar\omega\left(N + \frac{1}{2}\right) + \frac{\hbar c k_x E_y}{B} + \frac{mc^2 E_y^2}{2B^2}. \tag{37}$$

2.2 Laughlin's Gauge Argument

Suppose we change the flux adiabatically by a flux quantum $\Delta\Phi = hc/e$. Then in the exponent of the wavefunction Eq. (36) (which holds true for ground state and all excited states):

$$y + k_x l^2 - \frac{\Phi}{BL_x} \to y + \left(k_x - \frac{2\pi}{L_x}\right) l^2 - \frac{\Phi}{BL_x}. \tag{38}$$

The momenta of all the occupied states change by $2\pi/L_x$. Then according to Eq. (37), the energy changes per Landau level is

$$\Delta E = D\frac{\hbar c E_y}{B}\frac{2\pi}{L_x} = eE_y L_y = eV_y. \tag{39}$$

This exactly means we transfer one electron from one edge to another per Landau level.

On the other hand, according to Feynman-Hellman theorem:

$$\langle N, k_x, \Phi|\frac{\partial H}{\partial \Phi}|N, k_x, \Phi\rangle = \frac{\partial E_N}{\partial \Phi}. \tag{40}$$

Insert Eq. (35),

$$j_x = c\frac{\Delta E}{\Delta\Phi} = c\frac{neV_y}{hc/e} = \frac{ne^2}{h}V_y, \tag{41}$$

where n is the number of occupied Landau levels. Then we immediately obtain quantized Hall conductivity:

$$\sigma_{xy} = \frac{j_x}{V_y} = \frac{ne^2}{h}. \tag{42}$$

As magnetic field increases, the Landau level degeneracy D becomes larger, and fewer Landau levels will be occupied. Hence n decreases with magnetic field. This (partially) explains Figure 1. (In real experiments, the situation is more complicated. In fact, one needs not too weak disorder in the system in order to see good integer quantum Hall effect, which is beyond our previous discussion.)

References

[1] R. B. Laughlin. Quantized Hall conductivity in two dimensions. *Phys. Rev. B*, 23(10):5632–5633, 1981.

[2] R. B. Laughlin. Nobel Lecture: Fractional quantization. *Rev. Mod. Phys.*, 71(4):863–874, 1999.